PO

BS → BS

Example 3.1. Consider

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 5 \\ 2x_1 - 3x_2 + x_3 + x_5 = 3 \\ -x_1 + 2x_2 - x_3 + x_6 = 1 \end{cases}$$

$A\vec{x} = \vec{b}$

3×6

The initial tableau is given by
Tableau 1:

Augmented $[A \mid \vec{b}]$ $B_1^{-1}b$ $\vec{a}_4\ \vec{a}_5\ \vec{a}_6$

	x_1	x_2	x_3	$\boxed{x_4}$	$\boxed{x_5}$	$\boxed{x_6}$	**b**
x_4	1^*	1	−1	1	0	0	5
x_5	2	−3	1	0	1	0	3
x_6	−1	2	−1	0	0	1	1

A

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$B_1 Y_1 = A$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B_2^{-1}}$$

The current basic solution is $[0, 0, 0, 5, 3, 1]^T$ which is clearly feasible. Suppose we choose $a_{1,1}$ as our pivot element. Then after one pivot operation, we have
Tableau 2:

ERO

x_4 to leave
x_1 to enter

Basic may not be feasible

Y_2 basic $B_2^{-1}b$ $\vec{a}_1\ \vec{a}_5\ \vec{a}_6$

	$\boxed{x_1}$	x_2	x_3	x_4	$\boxed{x_5}$	$\boxed{x_6}$	**b**
x_1	1	1	−1	1	0	0	5
x_5	0	-5^*	3	−2	1	0	−7
x_6	0	3	−2	1	0	1	6

B_2^{-1}

$$B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$B_2 Y_2 = A$

We note that the current basic solution is $[5, 0, 0, 0, -7, 6]^T$ which is infeasible. Using the new $(2, 2)$ entry as pivot, we have
Tableau 3:

ERO

x_5 is leaving
x_2 is entering

Y_3 $\vec{a}_1\ \vec{a}_2\ \vec{a}_6$

	$\boxed{x_1}$	$\boxed{x_2}$	x_3	x_4	x_5	$\boxed{x_6}$	**b**
x_1	1	0	$-\frac{2}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{18}{5}$
x_2	0	1	$-\frac{3}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{7}{5}$
x_6	0	0	$-\frac{1}{5}^*$	$-\frac{1}{5}$	$\frac{3}{5}$	1	$\frac{9}{5}$

$B_3^{-1}b$

B_3^{-1}

$$B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$B_3 Y_3 = A$

The current basic solution is $[18/5, 7/5, 0, 0, 0, 9/5]^T$ and is feasible. Finally, let us eliminate the last slack variable x_6 by replacing it by x_3.
Tableau 4:

ERO

Y_4 $\vec{a}_1\ \vec{a}_2\ \vec{a}_3$

	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_1	1	0	0	1	−1	−2	0
x_2	0	1	0	1	−2	−3	−4
x_3	0	0	1	1	−2	−5	−9

$B_4^{-1}b$

B_4^{-1}

$$B_4 = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

$B_4 Y_4 = A$

The current basic solution is $[0, -4, -9, 0, 0, 0]^T$ which is infeasible and degenerate. Thus we see that one cannot choose the pivot arbitrarily. It has to be chosen according to some feasibility criterion.

There are three important observations that we should note here. First the pivot operations which amounts to *elementary row operations* on the tableaus, are being recorded in the tableaus at the columns that correspond to the slack variables. In the example above, one can easily check that Tableau i is obtained from Tableau 1 by pre-multiplying Tableau 1 by the matrix formed by the columns of x_4, x_5 and x_6 in Tableau i. In the tableaus, the inverse of these matrices are computed

Tableau 1

$$\left[\, R \,\middle|\, \underbrace{I}_{\substack{B\\ \text{invertible}}} \,\right] \begin{pmatrix} 0\\0\\0\\5\\3\\1 \end{pmatrix} = \begin{pmatrix} 5\\3\\1 \end{pmatrix}$$

A

BS

Starting BS

X_4, X_5, X_6 basic

X_1, X_2, X_3 nonbasic

Tableau 2

$X_1\ X_5\ X_6$ basic

$X_4\ X_2\ X_3$ nonbasic

$$\begin{array}{ccc} X_1 & X_5 & X_6 \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ERO's $\alpha \vec{r}_i \to \vec{r}_i$

$\vec{r}_i - \alpha \vec{r}_j \to \vec{r}_i$

$\vec{r}_2 - 2\vec{r}_1 \to \vec{r}_2$

−1	2	2	−2	1	0	0	10
0	−5	1	−2	1	0		−7

$\vec{r}_3 + \vec{r}_1 \Rightarrow \vec{r}_3$

nonbasic: x_2, x_3, x_4 basic: x_1, x_5, x_6

T2:

$$
\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \vec{b} \\
\hline
1 & 1 & -1 & 1 & 0 & 0 & 5 \\
0 & -5 & 3 & -2 & 1 & 0 & -7 \\
0 & 3 & -2 & 1 & 0 & 1 & 6
\end{array}
$$

basic: x_1, x_2, x_6 nonbasic: x_3, x_4, x_5

T3:

$$
\begin{array}{l|cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \vec{b} \\
\hline
\vec{r}_1 - \vec{r}_2 \to \vec{r}_1 & 1 & 0 & -2/5 & 3/5 & 1/5 & 0 & 18/5 \\
-\frac{1}{5}\vec{r}_2 \to \vec{r}_2 & 0 & 1 & -3/5 & 2/5 & -1/5 & 0 & 7/5 \\
\vec{r}_3 - 3\vec{r}_2 \to \vec{r}_3 & 0 & 0 & -1/5 & -1/5 & 3/5 & 1 & 9/5
\end{array}
$$

T_1 $\begin{array}{c} \\ \\ \\ x_4 \\ x_5 \\ x_6 \end{array}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 3 \\ 1 \end{pmatrix}$ is a ~~the~~ basic solution. $(\vec{a}_4, \vec{a}_5, \vec{a}_6) = I$.

T_2: $\begin{array}{c} x_1 \\ \\ \\ \\ x_5 \\ x_6 \end{array}\begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -7 \\ 6 \end{pmatrix}$ is also a solution

(i) Basic $(\vec{a}_1, \vec{a}_5, \vec{a}_6)$ invertible

(ii) not feasible.

T_3: $\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array}\begin{pmatrix} 18/5 \\ 7/5 \\ 0 \\ 0 \\ 0 \\ 9/5 \end{pmatrix}$ is also a solution

(i) Basic $(\vec{a}_1, \vec{a}_2, \vec{a}_6)$ invertible

(ii) feasible

T_1: $A = [R \mid I] = T_1$

T_2 $\underbrace{E_\ell \cdots E_1}\, A = E_\ell \cdots E_2 E_1 [R \mid I] = T_2$

$EA = E[R \mid I] = T_2 = [U \mid V]$

$\Rightarrow E = V$

T_3

$\cancel{=B} \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & * & * & * & \vec{e}_3 \end{pmatrix}$ $\quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underbrace{E_\ell \cdots E_1}_{E} A = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & * & * & * & \vec{e}_3 \end{pmatrix}$ $\quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$(\vec{a}_1\, \vec{a}_2 \cdots \vec{a}_6) = A = E^{-1} \overbrace{(\vec{e}_1\ \vec{e}_2 \times \cdots \times \vec{e}_3)}^{T_3}$

$= F\ (\vec{e}_1\ \vec{e}_2 \times \cdots \times \vec{e}_3) = (\vec{f}_1, \vec{f}_2, *, *, * \vec{f}_3)$

$= (\underbrace{\vec{a}_1, \vec{a}_2, \vec{a}_6}) (\vec{e}_1, \vec{e}_2 \cdots \vec{e}_3)$

$(\quad \overbrace{\vec{e}_1, \vec{e}_2, \vec{e}_3}^{I}) = A = B_3 (\vec{e}_1\ \vec{e}_2\ \boxed{\cdots \vec{e}_3})$ $\quad B_3^{-1}$

$A = B_3 \boxed{T_3}$

$\| $

T_3